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V Semester III B.Sc. Examination, March/April - 2021
(Semester Scheme) (Prior to 2015-16 Batch)
MATHEMATICS (Paper - VI)
Algebra and Calculus

Time : 3 Hours **Max. Marks : 80**

Instructions : *1) Section - A is compulsory.*

2) Answer any Five full questions from sections B and C choosing atleast two from each section.

3) All questions in sections B and C carry equal marks.

SECTION - A

1. Answer any ten questions. Each question carries two marks.

a) In a ring $(R, +, \bullet)$, show that $x(-y) = (-x)y = -(xy)\ \forall\, x, y \in R$.

b) Define Zero divisors of a ring. Find all the Zero divisors of $\{Z_8, \oplus_8, \otimes_8\}$.

c) Find all the units of Z[i].

d) If U is an ideal of R and $1 \in U$ then prove that U = R.

e) Prove that $U = \{5r / r \in \mathbb{Z}\}$ is a prime ideal of $(\mathbb{Z}, +, \bullet)$.

f) Write all the elements of the quotient ring $\frac{\mathbb{Z}}{I}$ where $I = 5\mathbb{Z}$.

g) If f is a homomorphism of a ring R into a ring R' then prove that $f(0) = 0'$, where 0 and 0' are Zeroes of R and R' respectively.

h) Show that $\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.

i) Find all zeroes of $x^4 + 2x^3 + x + 2$ over $\mathbb{Z}_3$.

j) Test the polynomial $P(x) = 7x^4 - 2x^3 + 6x^2 - 10x + 18$ for irreducibility over Q using Eisenstein's Criterion.

P.T.O.

k) If $P_1=\left\{0,\frac{1}{2},1,\frac{3}{2},2\right\}$, $P_2=\left\{0,\frac{1}{3},\frac{2}{3},1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2\right\}$ are two partitions of [0, 2] then find the common refinement of P_1 and P_2.

l) If $f(x)=3x\ \forall x\in[0,1]$, $P=\left\{0,\frac{1}{2},1\right\}$, find $L(P,f)$ and $U(P,f)$.

m) If $f:[a,b]\to R$ is bounded and P is a partition of [a, b] then show that $L(p,f)\le U(P,f)$.

n) Give an example of a bounded function which is not Riemann Integrable.

o) State second mean value theorem of Integral calculus.

SECTION - B

2. a) Prove that $\mathbb{Z}[\sqrt{2}]=\left\{a+b\sqrt{2}\,\middle|\,a,b\in\mathbb{Z}\right\}$ is a ring under usual addition and multiplication.

b) Show that $R_a=\{xa|x\in R|x.a=0\}$ is a subring of a ring R, where 'a' is the fixed element of R.

c) Prove that Z_n is an integral domain if and only if n is a prime.

3. a) Show that every finite integral domain is a field.

b) If S and T are two ideals of a commutative ring R then prove that S + T is also an ideal of R.

c) Define Kernel of a homomorphism $f: R\to R'$ and show that it is an ideal of a ring R.

4. a) Prove that the domain z of integers can be embedded isomorphically in the field of rational numbers.

b) Show that the correspondence $a+b\sqrt{7}\to a-b\sqrt{7}$ is an automorphism of the ring $\mathbb{Z}\left[\sqrt{7}\right]$.

c) Show that i) $3+\sqrt{2}$ and $5+4\sqrt{2}$ are associates in $Z\left[\sqrt{2}\right]$ ii) $2+\sqrt{5}$ and $2-\sqrt{5}$ are units in $Z\left[\sqrt{5}\right]$.

5. a) In the domain $Z\left[\sqrt{2}\right]=\left\{a+b\sqrt{2}\,\middle|\,a,b\in Z\right\}$ define norm of an element and prove that $N(\alpha\beta)=N(\alpha).N(\beta)\,\forall\,\alpha,\beta\in\mathbb{Z}\left[\sqrt{2}\right]$.

b) If f is a homomorphism from a ring R onto a ring R' with Kernel K then show that R|k is isomorphic to R'.

c) Show that $f(x)=x^3+x^2+x+1$ is reducible over the field $\mathbb{Z}_3$ and express it as a product of irreducible factors over $\mathbb{Z}_3$.

6. a) Prove that a polynomial function $f(x)$ is divisible by $(x-a)$ over a domain if and only if $f(a)=0$.

b) Find the GCD of $f(x)=x^{18}-1$ and $g(x)=x^{33}-1$ in Q[x], and express the G.C.D. as a linear combination of $f(x)$ and $g(x)$.

c) Test the equation $3x^3+x^2+x-2=0$ for rational roots.

SECTION - C

7. a) If f: $[a,b]\to$ R is Riemann Integrable over [a, b] then show that

$$m(b-a)\leq\int_a^b f(x)dx\leq M(b-a)$$

where m is infimum and M is supremum of $f(x)$ over [a, b]

b) Using Riemann Integration, show that $\int_0^1 f(x)dx=2$ where $f(x)=2x+1$.

c) Show that $f(x)=x^2$ is Riemann Integrable on [1, 2].

8. a) If f is a bounded function on $[a, b]$, for any two partitions P_1 and P_2 of $[a, b]$, such that $P_1 < P_2$ prove that $L(P_1, f) \leq L(P_2, f)$ and $U(P_1, f) > U(P_2, f)$.

   b) Show that a bounded function $f(x)$ is Riemann integrable on [a, b] iff there exists a number I such that for each $\varepsilon > 0$ there is partition P of $[a, b]$ such that $|U(P, f) - I| < \varepsilon$ and $|I - L(P, f)| < \varepsilon$.

   c) Show that every monotonic function is Riemann integrable on $[a, b]$.

9. a) If the function is defined in the interval [0, 1] as $f(x)=\frac{1}{n}$ for $\frac{1}{n+1}<x\leq\frac{1}{n}$, $n = 1, 2, 3, \ldots$ and $f(0) = 0$ then prove that $f(x)$ is Riemann integrable in [0, 1] and hence evaluate $\int_0^1 f(x)dx$.

   b) If $f : [a, b] \to R$ is continuous then Prove that there exist a point C in (a, b) such that $\int_a^b f(x)dx = f(c)(b-a)$.

   c) If $f : [a, b] \to R$ is continuous and $g'(x) = f(x)\ \forall x \in [a, b]$ then Prove that $\int_a^b f(x)dx = g(b) - g(a)$